Unsolvable Problems

Part One

Outline for Today

- ***Self-Reference Revisited***
  - Programs that compute on themselves.
- ***Self-Defeating Objects***
  - Objects “too powerful” to exist.
- ***The Fortune Teller***
  - Can you escape your fate?
- ***Why Do Programs Loop?***
  - … and can we eliminate loops?
- ***Undecidable Problems***
  - Something beyond the reach of algorithms.

Recap from Last Time

R and **RE**

- A language L is ***recognizable*** if there is a TM M with the following property:

$$\boldsymbol{\forall w \in \Sigma^*. \; (M \text{ accepts } w \leftrightarrow w \in L)}.$$

- That is, for any string w:
  - If $w \in L$, then M accepts w.
  - If $w \notin L$, them M does not accept w.
    - It might reject w, or it might loop on w.
- This is a “weak” notion of solving a problem.
- The class **RE** consists of all the recognizable languages.

R and **RE**

- A language L is ***decidable*** if there is a TM M with the following properties:

$$\forall w \in \Sigma^*.\ (M \text{ accepts } w \leftrightarrow w \in L).$$

$$M \text{ halts on all inputs.}$$

- That is, for any string w:
  - If $w \in L$, then M accepts w.
  - If $w \notin L$, then M rejects w.
- This is a “strong” notion of solving a problem.
- The class **R** consists of all the decidable languages.

The Universal TM

- The ***universal Turing machine***, denoted U_{TM}, is a TM with the following behavior: when run on a string $\langle M, w\rangle$, where M is a TM and w is a string, U_{TM} will

  … accept $\langle M, w\rangle$ if M accepts w,

  … reject $\langle M, w\rangle$ if M rejects w, and

  … loop on $\langle M, w\rangle$ if M loops on w.

- $\mathbf{A_{TM}}$ is the language recognized by the universal TM. This is the language

$$\mathbf{A_{TM} = \{ \langle \mathit{M}, \mathit{w} \rangle \mid \mathit{M} \text{ is a TM and } \mathit{M} \text{ accepts } \mathit{w} \}}$$

Self-Referential Programs

- Computing devices can compute on their own source code:

  Theorem: It is possible to construct TMs that perform arbitrary computations on their own source code.

- This allows us to write programs that work on their own source code.

New Stuff!

Part One: Self-Defeating Objects

A ***self-defeating object*** is an object whose essential properties ensure it doesn’t exist.

Question: Why is there no largest integer?

Answer: Because if n is the largest integer, what happens when we look at $n+1$?

Self-Defeating Objects

Theorem: There is no largest integer.

Proof sketch: Suppose for the sake of contradiction that there is a largest integer. Call that integer n.

Consider the integer $n+1$.

Notice that $n < n+1$.

But then n isn't the largest integer.

Contradiction!

An Important Detail

Careful – we're assuming what we're trying to prove!

Claim: There is a largest integer.

Proof: Assume x is the largest integer.

Notice that $x > x - 1$.

So there's no contradiction.

How do we know there's no contradiction? We just checked one case.

Self-Defeating Objects

- If you can show

  $$\textbf{\textit{x exists}} \rightarrow \bot$$

  then you know that *x* doesn't exist. (This is a proof by contradiction.)

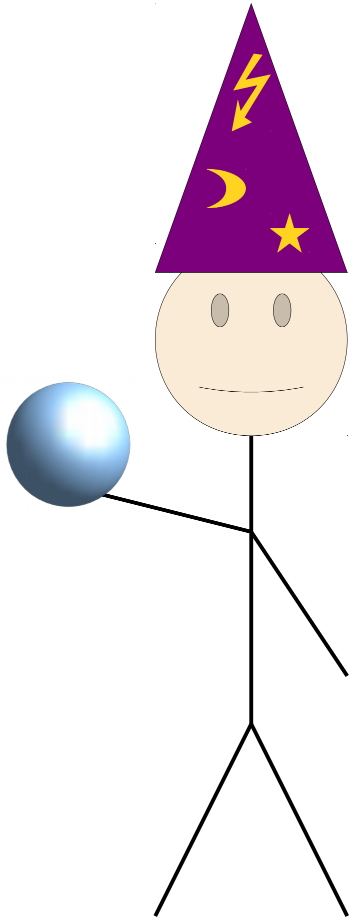
- If you can show

  $$\textbf{\textit{x exists}} \rightarrow \top$$

  you cannot conclude that *x* exists. (This is not a valid proof technique.)

Part Two: The Fortune Teller

The Fortune Teller



- A fortune teller appears who claims they can see into the future.
- For a nominal fee, the fortune teller will tell you anything you want to know about the future.

The Fortune Teller

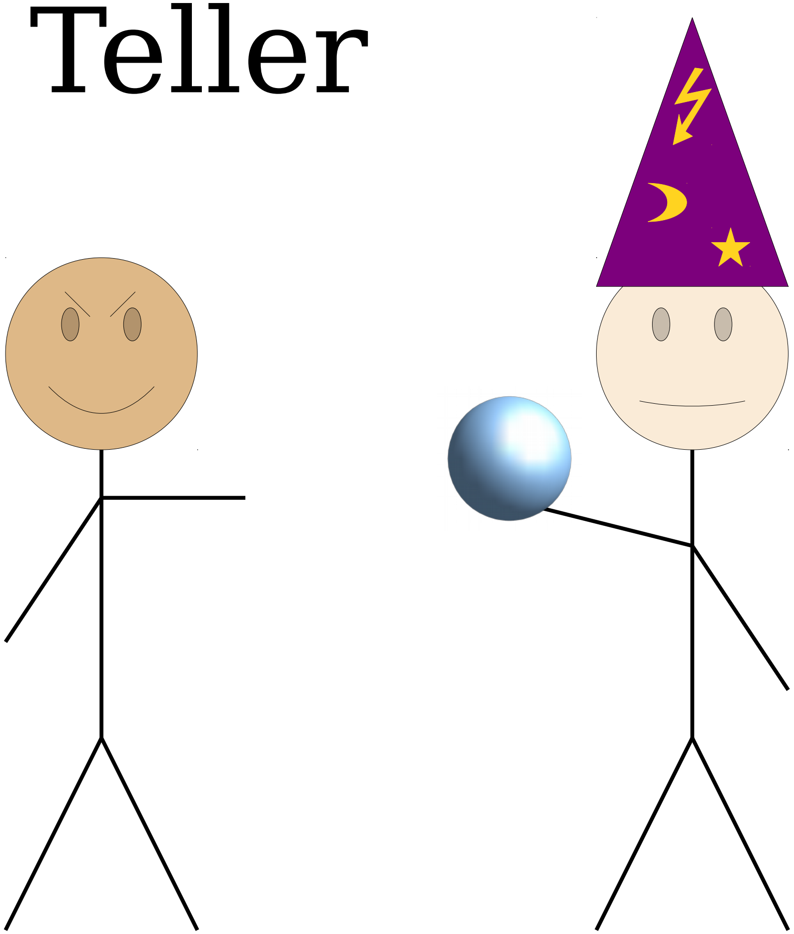
- One day, a trickster arrives. The trickster thinks the fortune teller is lying and can't really see the future.
- The trickster says the following:

"I have a yes/no question about the future. But before I ask my question, let's talk payment.

If you answer 'yes,' then I'll pay you $42.

If you answer 'no,' then I'll pay you $137."

- The fortune teller thinks for a moment, then agrees.



Trickster pays $42 if the fortune teller answers "yes."

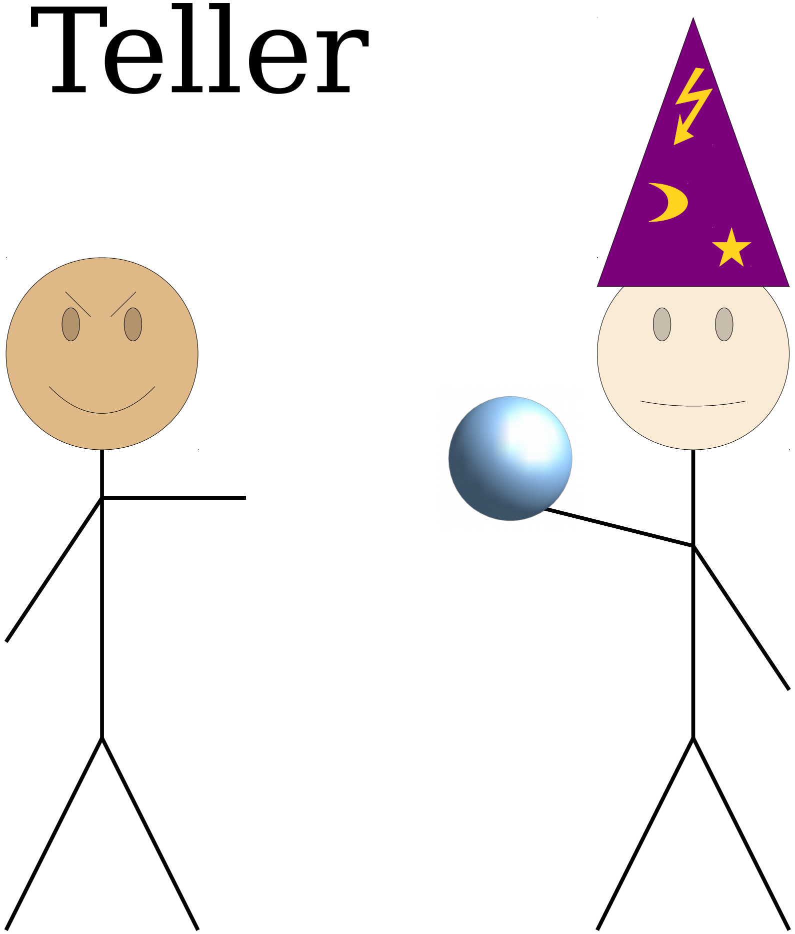
Trickster pays $137 if the fortune teller answers "no."

The Fortune Teller

- The trickster then asks this question:

  "Am I going to pay you $137?"

- The fortune teller is trapped!
- Why?



Trickster pays $42 if the fortune teller answers "yes."

Trickster pays $137 if the fortune teller answers "no."

The Fortune Teller

- The payment scheme the fortune teller agreed to means

Fortune Teller Says Yes ↔ ***Trickster Pays $42***.

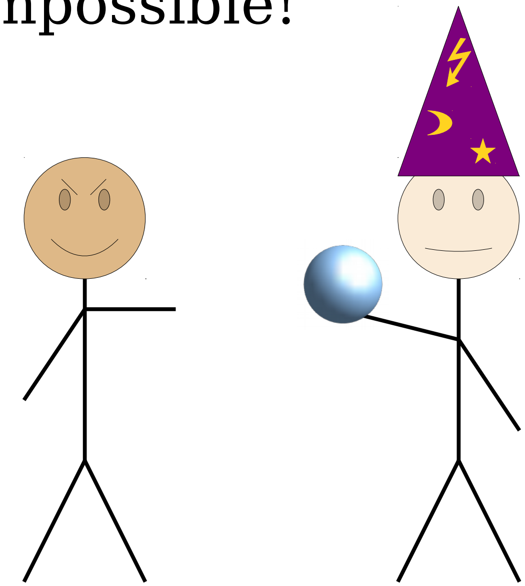
- The trickster’s question to the fortune teller means

Fortune Teller Says Yes ↔ ***Trickster Pays $137***.

- Putting this together, we get

Trickster Pays $137 ↔ ***Trickster Pays $42***.

- This is impossible!

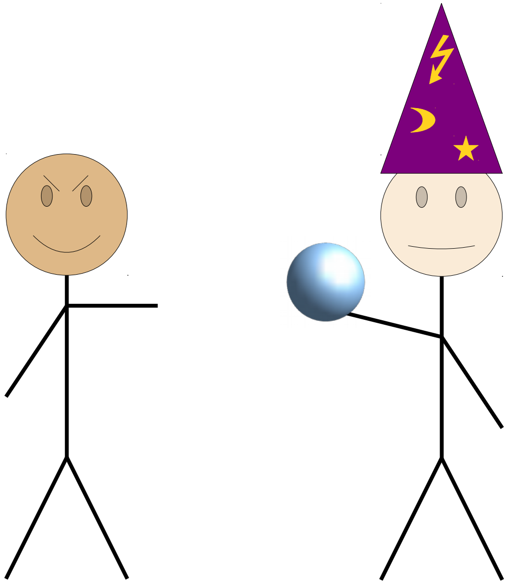


Trickster pays $42 if the fortune teller answers “yes.”

Trickster pays $137 if the fortune teller answers “no.”

The Fortune Teller

- The fortune teller is a self-defeating object.
- The trickster's strategy is to couple the fortune teller's behavior to what the future holds.
  - The trickster's behavior is chosen in advance to make the fortune teller's answer wrong.
- Therefore, the fortune teller can't answer all questions about all people in the future.



Trickster pays $42 if the fortune teller answers "yes."

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Part Three: Why Do Programs Loop?

Thoughts on Loops

- In practice, the programs we write sometimes go into infinite loops.
- In Theoryland, Turing machines are allowed to loop. This happens if they don't accept and don't reject.
- ***Question:*** Why are infinite loops possible?
- Or rather: are infinite loops an inherent part of computation, or are they some weird sort of "accident" in how we program computers?

Thoughts on Loops

- ***Theorem:*** The language A_{TM} is recognizable, but undecidable.
  - There's a *recognizer* for A_{TM} (specifically, the universal Turing machine U_{TM}).
  - It is impossible to build a *decider* for this language.
- Stated differently, there's a program we can write (a universal TM) that *has* to loop infinitely on some inputs.
- ***Goal:*** Prove this theorem, and explore its theoretical and philosophical implications.

A_{TM} Revisited

- As a refresher, the language A_{TM} is

  A_{TM} = { ⟨*M*, *w*⟩ | *M* is a TM and *M* accepts *w* }.

- The universal TM U_{TM} has the following behavior when given as input a TM *M* and a string *w*:
  - If *M* accepts *w*, then U_{TM} accepts ⟨*M*, *w*⟩.
  - If *M* rejects *w*, then U_{TM} rejects ⟨*M*, *w*⟩.
  - If *M* loops on *w*, then U_{TM} loops on ⟨*M*, *w*⟩.
- U_{TM} is a recognizer for A_{TM}, but because of that last case it's not a decider for A_{TM}.

A_{TM} Revisited

- As a refresher, the language A_{TM} is

  $\mathbf{A}_{\mathbf{TM}} = \{ \langle M, w \rangle \mid M$ is a TM and M accepts $w \}$.

- Given a TM M and a string w, a decider D for A_{TM} would need to have this behavior:
  - If M accepts w, then D **?** $\langle M, w \rangle$.
  - If M rejects w, then D **?** $\langle M, w \rangle$.
  - If M loops on w, then D **?** $\langle M, w \rangle$.
- This is basically the same set of requirements as U_{TM}, except for what happens if M loops on w.
- Our goal is to prove that there is no way to build a program that meets these requirements.

A_{TM} Revisited

- We can envision a decider for A_{TM} as a function

  ```
  bool willAccept(string fn, string input)
  ```

  that takes as input the source code of a function (`fn`) and a string representing an input to that function (`input`).
- It then does the following:
  - If `fn(input)` returns true, `willAccept(fn, input)` returns true.
  - If `fn(input)` returns false, `willAccept(fn, input)` returns false.
  - If `fn(input)` loops, then `willAccept(fn, input)` returns false.
- We're going to show it's impossible to write a function that actually does this. But for now, let's just explore what such a decider would do.

Why is A_{TM} Hard?

- ***Intuition:*** A decider for A_{TM} would be able to…
  - … determine whether the hailstone sequence terminates for any input. (Write a recognizer that runs the hailstone sequence, then feed it into the decider for A_{TM}.)
  - … solve other open math problems.
  - … and much, much more.
- In other words, this seemingly simple problem of “is this program going to terminate?” accidentally scoops up a bunch of other seemingly harder problems.

Time-Out for Announcements!

Preparing for the Final Exam

- We’ve posted a gigantic compendium of CS103 practice problems on the course website.
- You can search for problems based on the topics they cover, whether solutions are available, whether they’re ones we particularly like, and whether they were used on past exams.
- We’re currently putting together practice exams as well. You can expect those to be released on Friday.

Preparing for the Final Exam

- To emphasize: we definitely don’t expect you to do all of the practice problems. In fact, we recommend trying to focus on quality over quantity.
- Take some time to review what concepts or skills we flagged for you on the midterms and past assignments. Then, focus in on those skills in your preparation.
- As always, ***keep the TAs in the loop!*** Ask us questions if you have them, feel free to stop by office hours to discuss solutions, etc.

Back to CS103!

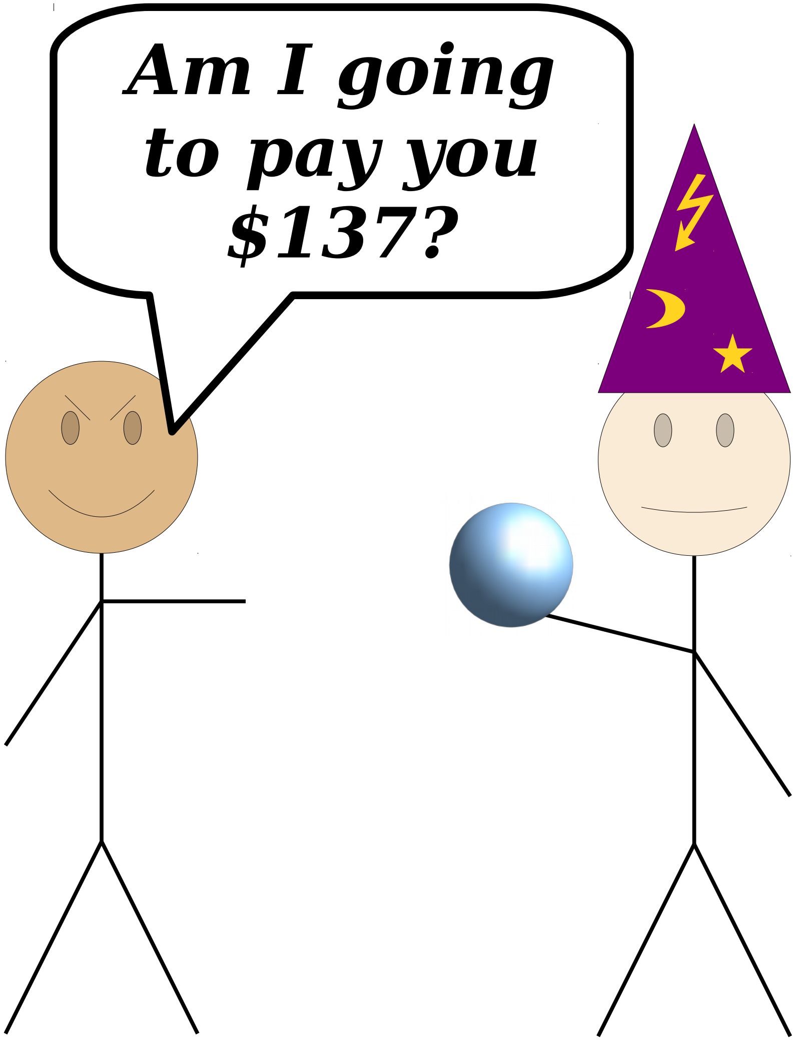
Part Four: Putting It All Together

To Recap

- We're assuming that, somehow, someone wrote a function

  ```
  bool willAccept(string function, string input);
  ```

  that takes the code of a function and an input to that function, then
  - returns true if `function(input)` returns true, and
  - returns false if `function(input)` doesn't return true.
- ***Goal:*** Show that this decider is "self-defeating;" its power is so great that it undermines itself.
- ***Idea:*** Convert the fortune teller story into a program.

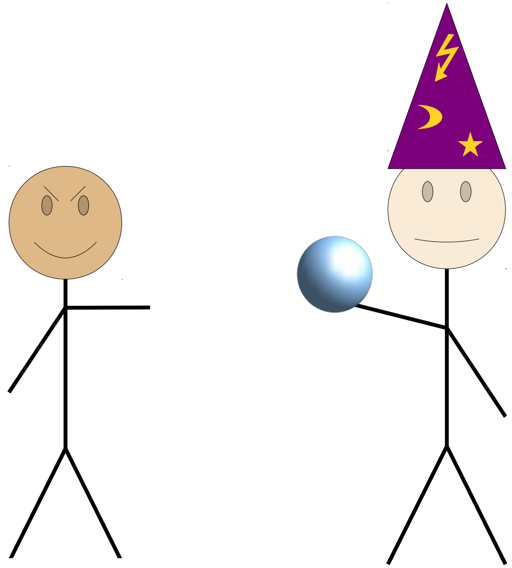


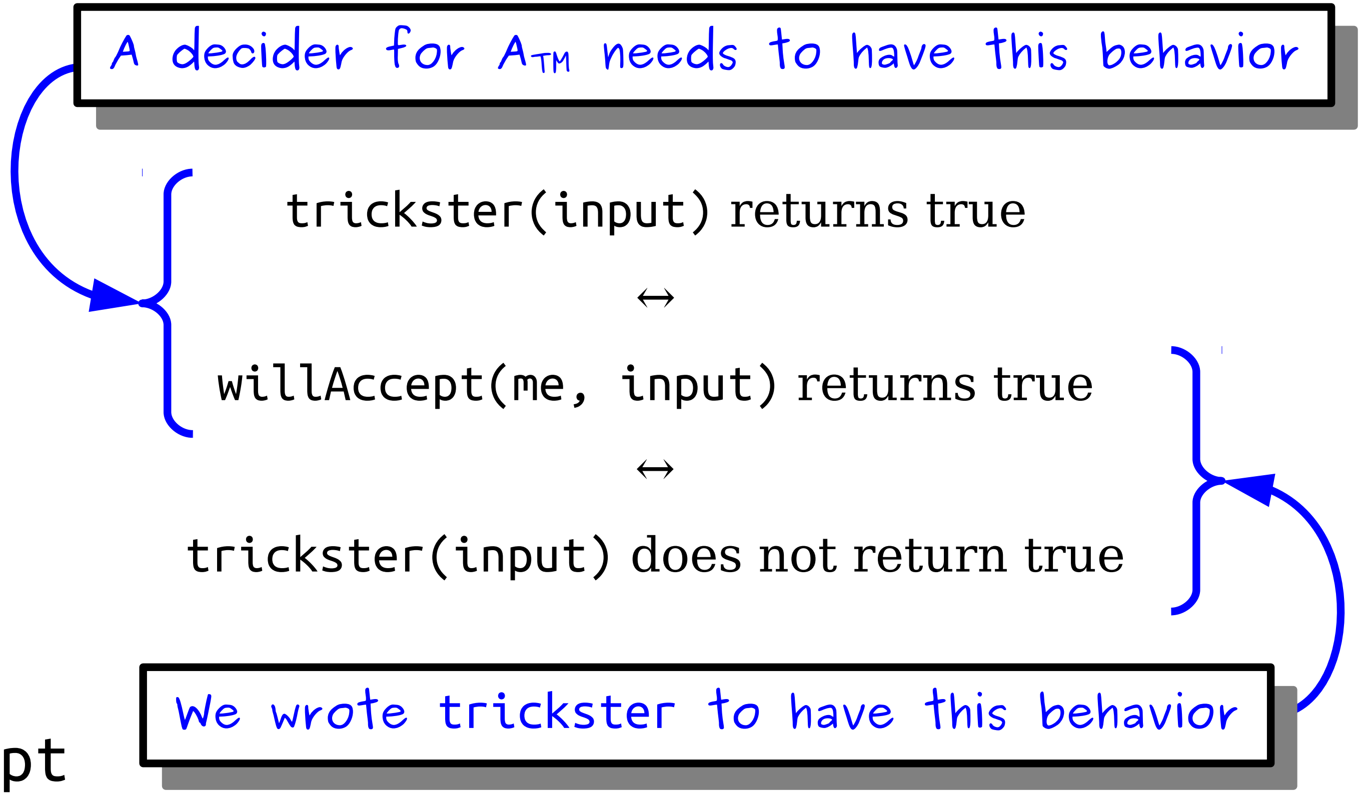
Trickster pays $42 if the fortune teller answers “yes.”

Trickster pays $137 if the fortune teller answers “no.”

```
bool willAccept(string function, string input) {
   // Returns true if function(input) returns true.
   // Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster */;
   return !willAccept(me, input);
}
```





A decider for A_{TM} needs to have this behavior

`trickster(input)` returns true

↔

`willAccept(me, input)` returns true

↔

`trickster(input)` does not return true

We wrote trickster to have this behavior

```
bool willAccept(string function, string input) {
   // Returns true if function(input) returns true.
   // Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster
   return !willAccept(me, input);
}
```

A self-defeating object.

Using that object against itself.

```
bool willAccept(string function, string input) {
   // Returns true if function(input) returns true.
   // Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster */;
   return !willAccept(me, input);
}
```

"The largest integer n."

"The integer $n + 1$."

Theorem: There is no largest integer.

Proof sketch: Suppose for the sake of contradiction that there is a largest integer. Call that integer n.

Consider the integer $n+1$.

Notice that $n < n+1$.

But then n isn't the largest integer.

Contradiction!

Theorem: $A_{TM} \notin \mathbf{R}$.

Proof: By contradiction; assume that $A_{TM} \in \mathbf{R}$. Then there is a decider D for A_{TM}. We can represent D as a function

```
bool willAccept(string function, string w);
```

that takes in the source code of a function `function` and a string `w`, then returns true if `function(w)` returns true and returns false otherwise. Given this, consider this function `trickster`:

```
bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}
```

Since `willAccept` decides A_{TM} and `me` holds the source of `trickster`, we know that

`willAccept(me, input)` returns true if and only if `trickster(input)` returns true.

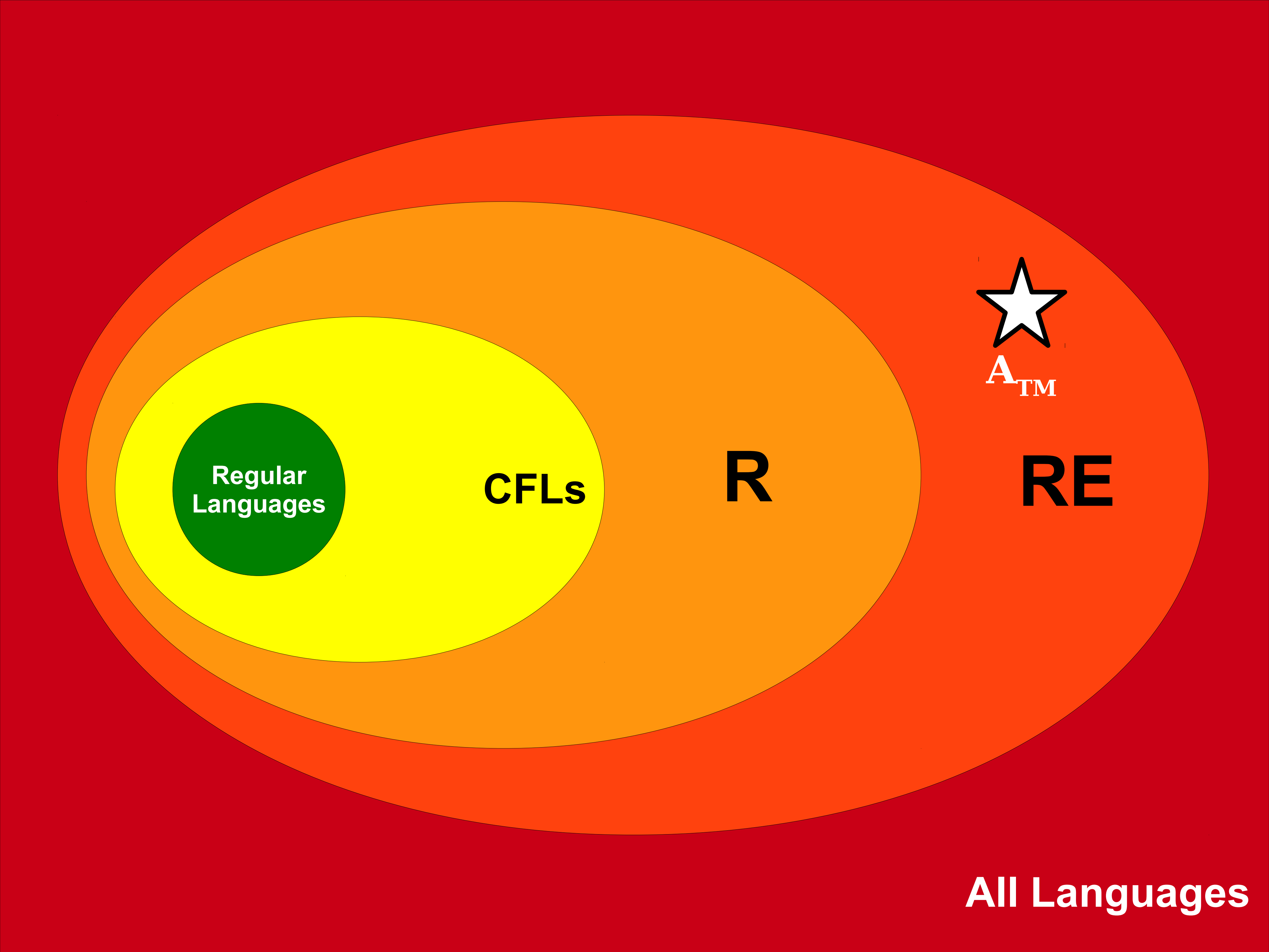
Given how `trickster` is written, we see that

`willAccept(me, input)` returns true if and only if `trickster(input)` doesn't return true.

This means that

`trickster(input)` returns true if and only if `trickster(input)` doesn't return true.

This is impossible. We've reached a contradiction, so our assumption was wrong and A_{TM} is undecidable. ■



A_{TM}
Regular Languages
CFLs
R
RE
All Languages

What Does This Mean?

- In one fell swoop, we've proven that
  - A_{TM} is ***undecidable***; there is no general algorithm that can determine whether a TM will accept a string.
  - **R** ≠ **RE**, because $A_{TM} \notin$ **R** but $A_{TM} \in$ **RE**.
- What do these three statements really mean? As in, why should you care?

$A_{TM} \notin \mathbf{R}$

- What exactly does it mean for A_{TM} to be undecidable?

Intuition: The only general way to find out what a program will do is to run it.

- As you'll see, this means that it's provably impossible for computers to be able to answer most questions about what a program will do.

$A_{TM} \notin \mathbf{R}$

- At a more fundamental level, the existence of undecidable problems tells us the following:

There is a difference between what is true and what we can discover is true.

- Given a TM *M* and a string *w*, one of these two statements is true:

M* accepts *w **_M_ does not accept _w_**

But since A_{TM} is undecidable, there is no algorithm that can always determine which of these statements is true!

R ≠ RE

- Because **R ≠ RE**, there is a difference between decidability and recognizability:

  In some sense, it is fundamentally harder to solve a problem than it is to check an answer.

- There are problems where, when the answer is “yes,” you can confirm it (run a recognizer), but where if you don’t have the answer, you can’t come up with it in a mechanical way (build a decider).

Next Time

- ***Why All This Matters***
  - Important, practical, undecidable problems.
- ***Intuiting RE***
  - What exactly is the class **RE** all about?
- ***Verifiers***
  - A totally different perspective on problem solving.
- ***Beyond RE***
  - Finding an impossible problem using very familiar techniques.